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Handout
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Lemma H1. *Let $\mathcal{H}$ be a Hilbert space with associated inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|_{\mathcal{H}}$. Then, we have*

$$\|x\|_{\mathcal{H}} = \sup_{\|g\|_{\mathcal{H}}=1} |\langle x, g\rangle|, \qquad \forall x \in \mathcal{H}.$$

Lemma H2. *Let $U : \mathcal{H} \to \mathcal{K}$ be a bounded linear operator from a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{K}$. Then, the operator norm given by*

$$|||U||| := \sup_{\|x\|_{\mathcal{H}}=1} \|Ux\|_{\mathcal{K}}$$

is finite. Furthermore, we have

$$\|Ux\|_{\mathcal{K}} \leq |||U||| \|x\|_{\mathcal{H}}, \qquad \forall x \in \mathcal{H}.$$

Lemma H3. *Let $\mathcal{H}$ be a Hilbert space and let $U : \ell^2(\mathbb{N}) \to \mathcal{H}$ be a bijective bounded linear operator. Then, we have the following:*

(a) U is invertible and the inverse U^{-1} is a bijective bounded linear operator.

(b) U^*, i.e., the adjoint of U, is a bijective bounded linear operator.

Lemma H4 (Massart's lemma)**.** *Let $\mathcal{F}$ be a finite class of functions $f\colon \mathcal{X} \subset \mathbb{R}^d \to \mathbb{R}$. Suppose that there exists a constant $C > 0$ such that $\sup_{f\in\mathcal{F}} \frac{1}{n}\sum_{i=1}^n f(x_i)^2 \leq C^2$, for all $x_1^n \subseteq \mathcal{X}$. Then, it holds that*

$$\mathcal{R}\left(\mathcal{F}\left(x_1^n\right)/n\right) \leq C\sqrt{\frac{2\log(|\mathcal{F}|)}{n}}.$$

Here, $\log(\cdot)$ is to the base e.

Lemma H5 (Ledoux–Talagrand contraction)**.** *Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be an L-Lipschitz function with $\phi(0) = 0$ and let $\mathcal{F}$ be a class of functions $f\colon \mathcal{X} \subset \mathbb{R}^d \to \mathbb{R}$. Let $\phi \circ \mathcal{F} := \{\phi \circ f \mid f \in \mathcal{F}\}$. Then,*

$$\mathcal{R}\left(\left(\phi \circ \mathcal{F}\right)\left(x_1^n\right)/n\right) \leq 2L\mathcal{R}\left(\mathcal{F}\left(x_1^n\right)/n\right).$$

Definition H6. *Let $d \in \mathbb{N}$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We define the inner product on $\mathbb{K}^d$ as*

$$\langle x, y\rangle := \begin{cases} \sum_{i=1}^d x_i y_i, & \text{if } \mathbb{K} = \mathbb{R}, \\ \sum_{i=1}^d x_i \overline{y_i}, & \text{if } \mathbb{K} = \mathbb{C}, \end{cases} \qquad x, y \in \mathbb{K}^d,$$

where $\overline{z}$ denotes complex conjugation. For $p \in [1, \infty)$, the p-norm on $\mathbb{K}^d$ is defined as

$$\|x\|_p := \left(\sum_{i=1}^{d} |x_i|^p \right)^{1/p}, \quad x \in \mathbb{K}^d.$$

The ∞-norm on $\mathbb{K}^d$ is defined as

$$\|x\|_\infty := \max_{1 \le i \le d} |x_i|, \quad x \in \mathbb{K}^d.$$

Lemma H7. *Let $d \in \mathbb{N}$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. It holds that*

$$|\langle x, y \rangle| \le \|x\|_1 \|y\|_\infty, \quad x, y \in \mathbb{K}^d.$$

Lemma H8. *It holds that*

$$\{x \mapsto \langle x, w \rangle \colon w \in \mathbb{R}^d, \|w\|_1 \le 1\} = \operatorname{conv}\left(\left\{ x \mapsto \langle x, w \rangle \colon w \in \bigcup_{k=1}^{d} \{e_k, -e_k\} \right\} \right),$$

where $\{e_k\}_{k=1}^d$ denotes the standard basis of $\mathbb{R}^d$, i.e.,

$$(e_k)_j = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{otherwise,} \end{cases} \quad j, k \in \{1, \dots, d\}.$$

Definition H9. *Let X be a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. An inner product on X is a map $\langle \cdot, \cdot \rangle \colon X \times X \to \mathbb{K}$ such that for all $x, x_1, x_2, y \in X$ and $\lambda \in \mathbb{K}$, the following properties hold:*

(i) $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$,

(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$,

(iii) $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$,

(iv) $\langle x, x \rangle \ge 0$, *with equality if and only if* $x = 0$.

When $\mathbb{K} = \mathbb{R}$, the complex conjugation in (ii) is superfluous.